PROBLEM SETS

Some of these exercises are from Weibel's K-theory book [3]:

http://www.math.rutgers.edu/~weibel/Kbook/Kbook.pdf

1. Problem set 1 (Monday): Lower K-groups and GL_nR

Exercise 1.1. Let M be a commutative monoid. Show that the following definitions of the *group completion* $\mathrm{Gr}(M)$ are equivalent.

i. $\mathrm{Gr}(M)$ is the free abelian group on generators $[m]$, for $m \in M$, modulo the subgroup generated by elements of the form $[m]+[n]-[m+n]$.

ii. $\mathrm{Gr}(M)$ is the set theoretic quotient of $M \times M$ by the relation

$$(m,n) \sim (m+p, n+p)$$

and operation induced from M.

Exercise 1.2. Compute the group completions $\mathrm{Gr}(M)$ when M is each of the following monoids:

i. $\mathbb{N}$ with sum,
ii. $\mathbb{N}\backslash\{0\}$ with product,
iii. the monoid of finite sets with disjoint union.

Exercise 1.3. Give an example of a group completion map $M \to \mathrm{Gr}(M)$ which is not injective. Can you think of a condition on M which ensures this map is injective?

The next exercise connects the definitions of the algebraic K_0 group and the topological K^0 group and is a result due to Swan.

Exercise 1.4. Consider the ring $\mathcal{C}(X,\mathbb{C})$ of continuous functions $X \to \mathbb{C}$ on a compact Hausdorff space X, and let $\eta : E \to X$ be a complex vector bundle. Show that there is an isomorphism

$$KU^0_{\mathrm{top}}(X) \cong K_0(\mathcal{C}(X,\mathbb{C})).$$

Hint: Show that the category of complex vector bundles over X is equivalent to the category $\mathbf{P}(\mathcal{C}(X,\mathbb{C}))$ of finitely generated projective modules over the ring $\mathcal{C}(X,\mathbb{C})$. Note that the set $\Gamma(E) = \{s : X \to E : \eta s = 1_X\}$ of global sections of η forms a projective $\mathcal{C}(X,\mathbb{C})$-module.

Similarly, we obtain $KO^0_{\mathrm{top}}(X) \cong K_0(\mathcal{C}(X,\mathbb{R}))$.

The following exercises are useful for understanding the definition of K_1. Let R be a ring. Let $GLR = \bigcup GL_nR$, where we regard GL_nR as the subset of $GL_{n+1}R$ consisting of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

We can identify GLR as the invertible infinite matrices that are the identity off a finite submatrix.

An *elementary matrix* is a matrix of the form $I + \alpha\epsilon_{i,j}$ where $\alpha \in R$, $i \neq j$, and $\epsilon_{i,j}$ is the matrix that is zero everywhere except the i,j spot, where it is 1. Let $E_nR < GL_nR$ and $ER < GLR$ be the subgroups generated by the elementary matrices.

Recall that a group is said to be *perfect* if it is its own commutator subgroup.

Exercise 1.5. For $n \geq 3$, show that E_nR is perfect.

For the next exercise we recall Whitehead's Lemma: if $A \in GL_nR$, the matrix

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

is in $E_{2n}R$. We can deduce this from the following sequence of row operations.

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \mapsto \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & I \\ -I & A^{-1} \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ -I & A^{-1} \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

Exercise 1.6. If $A, B \in GL_nR$, show that $\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{pmatrix}$ is a product of matrices of the form $\begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix}$. Conclude that ER is the commutator subgroup of GLR.

2. Problem set 2 (Tuesday): Scissors congruence

Exercise 2.1. Prove that two polygons in the Euclidean plane are scissors congruent if and only if they have the same area.

Exercise 2.2. Consider the upper-half plane model of the hyperbolic plane, together with its boundary. Here, the points are the points (x, y) in $\mathbb{R}^2$ such that $y \geq 0$, along with an additional point at infinity. The lines are those lines parallel to the y-axis (and including the point at infinity), together with the circles that are orthogonal to the x-axis. Translations parallel to the x-axis and homotheties around points on the x-axis are all isometries.

Give an example of two polygons which have the same area but are NOT scissors congruent.

Hint: "has vertices lying on the boundary" is a scissors congruence invariant.

Exercise 2.3. Let X be a geometry which is Euclidean, spherical, or hyperbolic (without vertices at infinity). Prove that $[P] = [Q]$ in the scissors congruence group $\mathcal{P}(X)$ if and only if P and Q are scissors congruent.

To check your proof, show it FAILS for the hyperbolic plane with vertices at infinity.

3. Problem set 3 (Tuesday): Plus construction and group completion

Recall that a group is said to be *perfect* if it is its own commutator subgroup.

Definition. Let X be a based connected CW complex and P a perfect normal subgroup of $\pi_1 X$. A map $X \to X^+$ is said to be *a plus construction relative to* P when all the following hold:

i. X^+ is a connected CW complex (which we base at the image of the base point of X).

ii. The map $\pi_1 X \to \pi_1 X^+$ is surjective with kernel P.

iii. The map $X \to X^+$ induces an isomorphism on homology for any local coefficient system on X^+.

The next exercise gives a construction; the codomain is usually called *the plus construction*.

Exercise 3.1. Let X be a connected CW complex with a perfect normal subgroup P. Form Y from X by attaching one 2-cell e_p for each element p of P along a chosen 1-cell representing p. Then $\pi_1 Y = (\pi_1 X)/P$.

- Show that $H_2 Y$ is isomorphic to the direct sum of $H_2 X$ and the free abelian group generated by the classes $[e_p]$ represented by the 2-cells e_p (for all $p \in P$).
- Show the class $[e_p]$ is in the image of the Hurewicz homomorphism $\pi_2 Y \to H_2 Y$.

Choose a representing map $S^2 \to Y$ for each p, and form Z by attaching 3-cells along these maps.

- Show that $X \to Z$ is a plus construction relative to P.

The following exercise gives the universal property of the plus construction.

Exercise 3.2. Let $f : X \to X^+$ be the plus construction from Exercise 3.1, and let

$$g : X \to Y$$

be any map such that P is in the kernel of $\pi_1 g$. Show that there is a map

$$h : X^+ \to Y$$

such that $h \circ f \simeq g$ and that h is unique up to homotopy. Show that if g is a plus construction relative to P, then h is a homotopy equivalence.

Let R be a ring. Recall from Exercises 1.5 and 1.6 of problem set 1 that

$$GLR = \bigcup GL_n R,$$

and $E_n R < GL_n R$ and $ER < GLR$ are the subgroups generated by the elementary matrices. For $n \geq 3$, $E_n R$ is perfect, and ER is the commutator subgroup of GLR.

For $BGLR$, we always take ER as the perfect normal subgroup of π_1 to form $BGLR^+$. For $BGL_n R$ ($n \geq 3$) we take the normal closure of $E_n R$ to form $BGL_n R^+$.

Exercise 3.3. Show that for the sequence of maps $BGL_n R^+ \to BGL_{n+1} R^+$, compatible with the maps $BGL_n R \to BGL_n R^+$ induced by the inclusions, the homotopy colimit, $\operatorname{hocolim} BGL_n R^+$, is homotopy equivalent to $BGLR^+$.

Similar to the definition of GLR, let $\Sigma_\infty = \cup \Sigma_n$ where where we regard Σ_n as as subset of Σ_{n+1} by regarding as a permutation on n elements as a permutation on $n+1$ elements by permuting the first n elements. The definition of A_∞ is similar.

Exercise 3.4. Recall from lecture the Barratt–Priddy–Quillen theorem, that $B\Sigma_\infty^+ \simeq QS^0$.

i. Show that $B\Sigma_\infty^+ \simeq \mathbb{RP}^\infty \times BA_\infty^+$.
ii. Show that the map $\pi_1^s \to K_1(\mathbb{Z})$, induced by the map $\Sigma_n \to GL_n(\mathbb{Z})$ taking each permutation to its permutation matrix, takes the generator $\eta \in \pi_1^s \cong \mathbb{Z}/2\mathbb{Z}$ to the element $-1 \in K_1(\mathbb{Z})$.

Definition. A topological monoid M is *grouplike* if $\pi_0 M$ is a group.

The map $M \to \Omega BM$ is sometimes called *group completion* because in the homotopy category of topological monoids, this map is initial for maps out of M into grouplike topological monoids. (This follows from the fact that when M is grouplike the map $M \to$

ΩBM is a weak equivalence.) A basic result about group completion is the following, which can be found in [1, App. Q] and [2]. In the statement, note that because ΩBM is grouplike, the images of elements of $\pi_0 M$ in $H_*(\Omega BM)$ are multiplicative units.

Theorem. *If $\pi_0 M$ is in the center of $H_* M$, then the canonical map*

$$H_* M[(\pi_0 M)^{-1}] \to H_*(\Omega BM)$$

is an isomorphism.

Exercise 3.5. Let $M = \coprod BGL_n R$ (for $n \geq 0$), a topological monoid under block sum of matrices. Show that $BGLR^+$ is homotopy equivalent to the zero component of ΩBM.

Exercise 3.6. Let $M = \coprod B\Sigma_n$ (for $n \geq 0$), a topological monoid under block sum of permutations. Show that $B\Sigma_\infty^+$ is homotopy equivalent to the zero component of ΩBM.

4. Problem set 4 (Thursday): The $S_\bullet$ construction and Q construction

Exercise 4.1. Let R be a ring. In the lectures we have shown that the Grothendieck group of projective R-modules (without any finitely generated condition imposed) is trivial. Show that the higher K-theory of the category of projective R-modules is trivial in any degree.

Exercise 4.2. Show the THH of the category of all R-modules, or of all projective R-modules, is trivial.

Exercise 4.3. Let $\mathscr{C}$ be an exact category. Show that $\pi_1 BQ\mathscr{C}$ is the free abelian group on isomorphism classes of objects of $\mathscr{C}$, modulo the relation $[A]+[C]=[B]$ for every exact sequence $A \to B \to C$.

Exercise 4.4. Let $\mathscr{C}$ be a Waldhausen category. This exercise considers Thomason's alternative definition of the K-theory space and shows that it is equivalent to Waldhausen's $S_\bullet$-construction.

Define a simplicial category $wT_\bullet\mathscr{C}$ whose objects at level n are sequences of cofibrations

$$A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n$$

in $\mathscr{C}$ and morphisms are maps $A_i \to A'_i$ making such diagrams commute that satisfy the condition that for every $i \leq j$ the induced map

$$A'_i \cup_{A_i} A_j \to A'_j$$

is a weak equivalence. Show the realizations of the bisimplicial sets $N_\bullet wS_\bullet\mathscr{C}$ and $N_\bullet wT_\bullet\mathscr{C}$ are equivalent.

Hint: Consider an intermediate simplicial category $wT_\bullet^+\mathscr{C}$, which adds in the data of quotients to the objects of the categories $T_n\mathscr{C}$. Then show that the realization of its nerve is equivalent to both realizations bisimplicial sets we are considering. So we get the desired equivalence via a zig-zag.

Exercise 4.5. Prove the "Swallowing lemma": Suppose $\mathscr{A}$ is a subcategory of $\mathscr{B}$. Let $\mathscr{A}_n\mathscr{B}$ be the simplicial category with objects n-chains of maps in $\mathscr{A}$ and morphisms given by diagrams with vertical maps in $\mathscr{B}$. Then $\mathscr{A}_\bullet\mathscr{B}$ is a simplicial category. Show that the inclusion of bisimplicial sets $N_\bullet\mathscr{B} \to N_\bullet\mathscr{A}_\bullet\mathscr{B}$ is a weak equivalence on geometric realization.

5. Problem set 5 (Thursday-Friday): Cyclic nerves and THH

Exercise 5.1. Show that if A is non-commutative, the trace

$$M_n(A) \to A$$

is not invariant under conjugation, $\mathrm{tr}(PAP^{-1}) \neq \mathrm{tr}(A)$. Show that this is corrected when we pass to $HH_0(A) = A/(ab = ba)$.

The resulting map $M_n(A) \to HH_0(A)$ is called the *Hattori–Stallings trace*.

Exercise 5.2. Let A be a ring and let $M_n(A)$ be its ring of $n \times n$ matrices. Define the *multitrace* by

$$M_n(A)^{\otimes(k+1)} \to A^{\otimes(k+1)}$$
$$A^0 \otimes A^1 \otimes \cdots \otimes A^k \mapsto \sum_{i_0,i_1,\ldots,i_k} A^0_{i_0 i_1} \otimes A^1_{i_1 i_2} \otimes \cdots \otimes A^k_{i_k i_0}.$$

Check that this is a map of chain complexes (or simplicial abelian groups)

$$N^{\mathrm{cyc}}_\bullet M_n(A) \to N^{\mathrm{cyc}}_\bullet A$$

that on H_0 takes each matrix to its Hattori–Stallings trace. This map is an equivalence

$$HH_*(M_n(A)) \to HH_*(A),$$

and is useful in understanding the Dennis trace.

Exercise 5.3. A *category enriched in abelian groups* $\mathscr{C}$ consists of objects $a, b, \ldots$, abelian groups $\mathscr{C}(a,b)$ for each pair of objects a and b, and composition maps

$$\mathscr{C}(a,b) \otimes \mathscr{C}(b,c) \to \mathscr{C}(a,c)$$

that are associative and unital. Note we are writing our compositions from left to right, which is the opposite of the usual convention for composition of functions.

i. Check that if $\mathscr{C}$ has one object, this is the same thing as a ring.
ii. Check that if A is a ring, the category of left A-modules ${}_A\mathrm{Mod}$ can be enriched in abelian groups, taking the morphisms to be the abelian groups $\mathrm{Hom}_A(M,N)$ of A-linear maps.
iii. Explain how the ring A sits inside the category ${}_A\mathrm{Mod}$. Does the matrix ring $M_n(A)$ sit inside ${}_A\mathrm{Mod}$?

Exercise 5.4. A *functor of spectral categories* $F : \mathscr{C} \to \mathscr{D}$ is a function on objects, $F : \mathrm{ob}\mathscr{C} \to \mathrm{ob}\mathscr{D}$, and maps of spectra

$$F : \mathscr{C}(a,b) \to \mathscr{D}(F(a),F(b))$$

that commute with composition and the identity. Check that such a functor induces a map of cyclic bar constructions $\mathrm{THH}(\mathscr{C}) \to \mathrm{THH}(\mathscr{D})$.

Exercise 5.5. Let $\mathscr{C}$ be any spectral category, $a \in \mathrm{ob}\mathscr{C}$ any object, and let $A = \mathscr{C}(a,a)$ be the corresponding ring spectrum of maps from a to itself.

i. Show that there is a spectral functor

$$\mathscr{C} \to {}_A\mathrm{Mod} \tag{5.6}$$

defined by sending b to $\mathscr{C}(a,b)$.

ii. If $\mathscr{C}$ is the category of A-modules, explain why the functor in (5.6) is a pointwise equivalence of spectral categories.

In other words, it is a bijection on the objects and gives an equivalence on the mapping spectra.

If you prefer, instead prove the corresponding statement for rings and categories enriched in abelian groups.

Exercise 5.7.

i. Recall the proof that the nerve sends natural transformations $F \Rightarrow G$ of functors $F, G\colon \mathscr{C} \to \mathscr{D}$ to homotopies of maps $N_\bullet\mathscr{C} \to N_\bullet\mathscr{D}$. Show adjunctions go to homotopy equivalences.

ii. Show that the cyclic nerve sends natural **isomorphisms** $F \cong G$ of functors

$$F, G\colon \mathscr{C} \to \mathscr{D}$$

to homotopies. Equivalently, show that the cyclic nerve takes equivalences of categories to homotopy equivalences.

You can do this either directly or using a Dennis–Waldhausen–Morita argument.

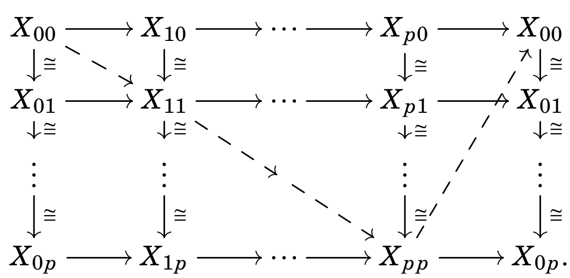
iii. Give an example of an adjunction of categories for which the cyclic nerves are not equivalent to each other.

Exercise 5.8. Let $\mathscr{C}$ be any category. Consider the bisimplicial set $i_\bullet N_\bullet^{\mathrm{cyc}}\mathscr{C}$ whose (p,q)th level is $(p+1)\times q$ grids of maps of the form

$$\begin{array}{ccccccccc}
X_{00} & \longrightarrow & X_{10} & \longrightarrow & \cdots & \longrightarrow & X_{p0} & \longrightarrow & X_{00} \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong \\
X_{01} & \longrightarrow & X_{11} & \longrightarrow & \cdots & \longrightarrow & X_{p1} & \longrightarrow & X_{01} \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong \\
X_{0q} & \longrightarrow & X_{1q} & \longrightarrow & \cdots & \longrightarrow & X_{pq} & \longrightarrow & X_{0q}.
\end{array}$$

The cyclic nerve $N_\bullet^{\mathrm{cyc}}\mathscr{C}$ includes into the diagonal of this bisimplicial set by making all of the vertical maps into identity maps. Prove that this is gives an equivalence on realizations. (The argument is similar to that of the swallowing lemma.)

Show that we can form an explicit inverse by taking each $p\times p$ grid to the sequence of maps illustrated by the dashed lines below:



If we include the nerve of isomorphisms $i_\bullet\mathscr{C}$ into this bisimplicial set and then apply this explicit inverse, we get the map taking $(f_1,\dots,f_q)$ to $(f_1,\dots,f_q,f_q^{-1}\cdots f_1^{-1})$. This can be used to show that the two different definitions of the Dennis trace we encountered agree with each other.

6. Problem set 6 (Friday): Group homology and higher scissors congruence

Exercise 6.1. Let G be a discrete group and let A be an abelian group with a left action of G through homomorphisms, i.e. a $\mathbb{Z}[G]$-module. Recall that group homology $H_n(G;A)$ is defined as $\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},A)$. In other words, we may calculate group homology by taking a projective resolution of A as a $\mathbb{Z}[G]$-module, applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}[G]} (-)$, and taking homology.

i. Explain why 0th homology $H_0(G;A)$ is isomorphic to the *coinvariants* A_G, the abelian group formed from A by applying the relation $a \sim ga$ for all $a \in A$ and $g \in G$. Conclude that the scissors congruence group $\mathcal{P}(X,G)$ is isomorphic to the homology group $H_0(G;\mathcal{P}(X,1))$, or to $H_0(G/N;\mathcal{P}(X,N))$ for any normal subgroup $N \leq G$.

ii. Consider $\mathbb{Q}$ as an abelian group under addition. Show that its group homology is a $\mathbb{Z}$ in degree 0, a $\mathbb{Q}$ in degree 1, and is zero in all higher degrees. (You might want to compute the group homology of $\mathbb{Z}$ first and recall that homology commutes with filtered colimits.)

iii. Building on the previous exercise, suppose that V is a rational vector space, considered as a group under addition. Prove that its group homology with $\mathbb{Z}$ coefficients is the exterior algebra

$$H_*(V;\mathbb{Z}) \cong \Lambda_*(V).$$

iv. Define the polytope group to be $\mathrm{Pt}(X) = \mathcal{P}(X,1)$. Prove that in the case of the Euclidean line, we have a short exact sequence

$$0 \to \mathrm{Pt}(E^1) \to \bigoplus_{\mathbb{R}} \mathbb{Z} \to \mathbb{Z} \to 0.$$

v. Recall that short exact sequences of coefficient groups $0 \to A \to B \to C \to 0$ induce long exact sequences on group homology $H_*(G;-)$. Use this to compute the translational scissors congruence group $\mathcal{P}(E^1,T(1))$, where $T(1) \cong \mathbb{R}$ is the group of translations of E^1 (as a discrete group).

We could have obtained the same answer by a more direct, elementary argument! But this approach also tells us the higher homology groups as well.

Exercise 6.2. Suppose X is n-dimensional Euclidean geometry E^n, $E(n)$ is the group of Euclidean isometries, and $T(n) \cong \mathbb{R}^n$ is the subgroup of translations. Show that the group homology of $T(n)$ and of $E(n)$ is a rational vector space in every degree. (Hint: use the first to prove the second!)

This can be used to show that the Euclidean scissors congruence groups $\mathcal{P}(E^n)$ are rational vector spaces. The rationality of the spherical groups $\mathcal{P}(S^n)$ and the hyperbolic groups $\mathcal{P}(H^n)$ is an open problem.

Exercise 6.3. The "Center Kills" Lemma states that if $g \in Z(G)$ is an element in the center of G and g acts on the coefficients A by multiplication by $r \in \mathbb{Z}$, then the homology groups $H_*(G;A)$ are all $(r-1)$-torsion. Use this to argue that

$$H_*(O(2n-1);\mathbb{Q}^t) = 0.$$

Here $\mathbb{Q}^t$ is the $\mathbb{Z}[O(2n-1)]$-module given by the rationals $\mathbb{Q}$, with $g \in O(2n-1)$ acting by $+1$ if g preserves orientation and -1 if g reverses orientation.

On the other hand, $H_*(O(2n);\mathbb{Q}^t)$ is not zero. What changes about the argument here? (This is related to the fact that Dehn invariants only exist for subspaces of even codimension.)

Exercise 6.4. Compute $H_*(SO(2);\mathbb{Q})$, where $SO(2)$ is considered as a discrete group. (The answer is completely different from the one you may have seen in a unit on characteristic classes.)

Exercise 6.5. If $H \leq G$ is a subgroup and A is a $\mathbb{Z}[H]$-module, we can form the induced module

$$G \otimes_H A := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A \cong \bigoplus_{G/H} A.$$

The Shapiro Lemma states that the homology of this induced module agrees with the homology of A:

$$H_*(G; G \otimes_H A) \cong H_*(H; A).$$

Use this to compute the homology of $O(2)$ with coefficients in $O(2) \otimes_{SO(2)} \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q}$. Can you recover $H_*(O(2); \mathbb{Q})$ and $H_*(O(2); \mathbb{Q}^t)$ from this?

These groups turn out to be an important part of the picture of higher scissors congruence: the higher scissors congruence groups of the plane E^2 are given by

$$\mathcal{P}_m(E^2) \cong H_{m+2}(E(2); \mathbb{Q}^t) \;/\; H_{m+2}(O(2); \mathbb{Q}^t).$$

References

[1] Eric M. Friedlander and Barry Mazur, *Filtrations on the homology of algebraic varieties*, Mem. Amer. Math. Soc. **110** (1994), no. 529, x+110, With an appendix by Daniel Quillen. MR 1211371 (95a:14023) (cit. on p. 4).

[2] D. McDuff and G. Segal, *Homology fibrations and the "group-completion" theorem*, Invent. Math. **31** (1975/76), no. 3, 279–284. MR 0402733 (53 #6547) (cit. on p. 4).

[3] Charles A. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory. MR 3076731 (cit. on p. 1).